Exercise 3.4.3

Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0$, $f(x_0^-) = \alpha$ and $f(x_0^+) = \beta$] and df/dx is piecewise smooth.

(a) Determine the Fourier sine series of df/dx in terms of the Fourier cosine series coefficients of $f(x)$.

(b) Determine the Fourier cosine series of df/dx in terms of the Fourier sine series coefficients of $f(x)$.

Solution

Part (a)

Since $f(x)$ is continuous, it's piecewise smooth on the interval $0 \leq x \leq L$, so it has a Fourier cosine series representation.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

The coefficients are known to be

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx$$
$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx.$$

df/dx is expected to be a series of sines; because it's also piecewise smooth, it has a Fourier sine series representation.

$$\boxed{\frac{df}{dx} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}}$$

The aim is to determine B_n. Multiply both sides of equation by $\sin \frac{p\pi x}{L}$, where p is an integer,

$$\frac{df}{dx} \sin \frac{p\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\begin{aligned} \int_0^L \frac{df}{dx} \sin \frac{p\pi x}{L}\, dx &= \int_0^L \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx \\ &= \sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx \end{aligned}$$

Since the sine functions are orthogonal, this integral on the right is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$\int_0^L \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx = B_n \int_0^L \sin^2 \frac{n\pi x}{L}\, dx$$

Evaluate the integral on the right.

$$\int_0^L \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx = B_n \left(\frac{L}{2}\right)$$

Solve for B_n.

$$\begin{aligned}
B_n &= \frac{2}{L} \int_0^L \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx \\
&= \frac{2}{L} \left(\int_0^{x_0^-} \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx + \int_{x_0^+}^L \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx \right) \\
&= \frac{2}{L} \left[f(x) \sin \frac{n\pi x}{L} \bigg|_0^{x_0^-} - \int_0^{x_0^-} f(x) \frac{d}{dx} \left(\sin \frac{n\pi x}{L} \right) dx + f(x) \sin \frac{n\pi x}{L} \bigg|_{x_0^+}^L - \int_{x_0^+}^L f(x) \frac{d}{dx} \left(\sin \frac{n\pi x}{L} \right) dx \right] \\
&= \frac{2}{L} \left[f(x_0^-) \sin \frac{n\pi x_0^-}{L} - f(x_0^+) \sin \frac{n\pi x_0^+}{L} - \int_0^L f(x) \frac{d}{dx} \left(\sin \frac{n\pi x}{L} \right) dx \right] \\
&= \frac{2}{L} \left[\alpha \sin \frac{n\pi x_0}{L} - \beta \sin \frac{n\pi x_0}{L} - \int_0^L f(x) \left(\frac{n\pi}{L} \cos \frac{n\pi x}{L} \right) dx \right] \\
&= \frac{2}{L} (\alpha - \beta) \sin \frac{n\pi x_0}{L} - \frac{n\pi}{L} \left[\frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx \right]
\end{aligned}$$

Therefore,

$$\boxed{B_n = \frac{2}{L} (\alpha - \beta) \sin \frac{n\pi x_0}{L} - \frac{n\pi}{L} A_n.}$$

Part (b)

Since $f(x)$ is continuous, it's piecewise smooth on the interval $0 \leq x \leq L$, so it has a Fourier sine series representation.

$$f(x) = \sum_{n=1}^\infty B_n \sin \frac{n\pi x}{L}$$

The coefficients are known to be

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

df/dx is expected to be a series of cosines; because it's also piecewise smooth, it has a Fourier cosine series representation.

$$\boxed{\frac{df}{dx} = A_0 + \sum_{n=1}^\infty A_n \cos \frac{n\pi x}{L}} \tag{1}$$

The aim is to determine A_0 and A_n. Start by integrating both sides with respect to x from 0 to L.

$$\begin{aligned}
\int_0^L \frac{df}{dx}\, dx &= \int_0^L \left(A_0 + \sum_{n=1}^\infty A_n \cos \frac{n\pi x}{L} \right) dx \\
&= A_0 \int_0^L dx + \sum_{n=1}^\infty A_n \underbrace{\int_0^L \cos \frac{n\pi x}{L}\, dx}_{=0}
\end{aligned}$$

Evaluate the integral on the right.

$$\int_0^L \frac{df}{dx}\,dx = A_0(L)$$

Solve for A_0.

$$\begin{aligned}
A_0 &= \frac{1}{L}\int_0^L \frac{df}{dx}\,dx \\
&= \frac{1}{L}\left(\int_0^{x_0^-} \frac{df}{dx}\,dx + \int_{x_0^+}^L \frac{df}{dx}\,dx\right) \\
&= \frac{1}{L}\left[f(x_0^-) - f(0) + f(L) - f(x_0^+)\right]
\end{aligned}$$

Therefore,

$$\boxed{A_0 = \frac{1}{L}\left[\alpha - \beta + f(L) - f(0)\right].}$$

To obtain A_n, multiply both sides of equation (1) by $\cos\frac{p\pi x}{L}$, where p is an integer,

$$\frac{df}{dx}\cos\frac{p\pi x}{L} = A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\begin{aligned}
\int_0^L \frac{df}{dx}\cos\frac{p\pi x}{L}\,dx &= \int_0^L \left(A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)dx \\
&= A_0 \underbrace{\int_0^L \cos\frac{p\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx
\end{aligned}$$

Because the cosine functions are orthogonal, this second integral on the right is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero value.

$$\int_0^L \frac{df}{dx}\cos\frac{n\pi x}{L}\,dx = A_n \int_0^L \cos^2\frac{n\pi x}{L}\,dx$$

Evaluate the integral on the right.

$$\int_0^L \frac{df}{dx}\cos\frac{n\pi x}{L}\,dx = A_n\left(\frac{L}{2}\right)$$

Solve for A_n.

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L \frac{df}{dx}\cos\frac{n\pi x}{L}\,dx \\
&= \frac{2}{L}\left(\int_0^{x_0^-} \frac{df}{dx}\cos\frac{n\pi x}{L}\,dx + \int_{x_0^+}^L \frac{df}{dx}\cos\frac{n\pi x}{L}\,dx\right) \\
&= \frac{2}{L}\left[f(x)\cos\frac{n\pi x}{L}\bigg|_0^{x_0^-} - \int_0^{x_0^-} f(x)\frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right)dx + f(x)\cos\frac{n\pi x}{L}\bigg|_{x_0^+}^L - \int_{x_0^+}^L f(x)\frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right)dx\right] \\
&= \frac{2}{L}\left[f(x_0^-)\cos\frac{n\pi x_0^-}{L} - f(0) + f(L)\cos n\pi - f(x_0^+)\cos\frac{n\pi x_0^+}{L} - \int_0^L f(x)\frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right)dx\right]
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
A_n &= \frac{2}{L}\left[\alpha\cos\frac{n\pi x_0}{L} - f(0) + f(L)(-1)^n - \beta\cos\frac{n\pi x_0}{L} - \int_0^L f(x)\left(-\frac{n\pi}{L}\sin\frac{n\pi x}{L}\right)dx\right] \\
&= \frac{2}{L}\left[(\alpha-\beta)\cos\frac{n\pi x_0}{L} + f(L)(-1)^n - f(0)\right] + \frac{n\pi}{L}\left[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right]
\end{aligned}$$

Therefore,

$$\boxed{A_n = \frac{2}{L}\left[(\alpha-\beta)\cos\frac{n\pi x_0}{L} + f(L)(-1)^n - f(0)\right] + \frac{n\pi}{L}B_n.}$$